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Scaling behavior of a directed sandpile automata with random defects

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In a recent paper [B. Tadić, U. Nowak, K. D. Usadel, R. Ramaswamy, and S. Padlewski, Phys. Rev. A **45**, 8536 (1992)], the directed two-dimensional sandpile is modified to include a concentration p of randomly placed holes. For $p>0$, they observe a characteristic cutoff size for the duration (and mass) of avalanches. This breaks the formal power-law behavior that is a signature of self-organized criticality. The scaling of these characteristic sizes with the concentration of defects is observed empirically to follow a power law, and the exponents are numerically determined. In this Brief Report, it is shown that a previously described mean-field approximation can account for the observed exponents.

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The directed two-dimensional (2D) sandpile of Dhar and Ramaswamy [1] is modified by Tadić *et al.* [2], by introducing a concentration p of randomly placed defects (sites called "holes" which are not permitted to topple). The authors numerically estimate exponents μ_x (respectively, μ_m) which describe the scaling of characteristic duration x (respectively, mass m) of avalanches as a function of p, but do not provide any theoretical arguments to explain the observed values ($\mu_x = 1.01 \pm 0.02$, $\mu_m = 1.52 \pm 0.01$). The purpose of this Brief Report is to show that these values are consistent with a simple model introduced by Alstrøm [3] which treats the avalanches as a stochastic branching process.

In the directed 2D sandpile automata, each site on a triangular lattice has a resting value of 0 or 1. A site is chosen randomly and one is added to its value; if the resulting value is larger than one, then the site is said to "topple." In that case, its value is reduced by two, and both of its downstream neighbors are incremented by one. If any of the incremented sites now have a value larger than one, then they topple, and so on down the lattice. The duration of the avalanche is the number of time steps (equivalently, the downstream distance) until no more sites topple, and the mass is the total number of toppled sites.

Each toppled site causes either 0, 1, or 2 of its downstream sites to topple, and in the mean-field approximation, fixed probabilities are assigned to each case. Further, all the downstream sites are treated independently (neglecting geometical properties of the two-dimensional lattice), and it is assumed that each of these sites can cause 0, 1, or 2 of its downstream sites to topple, with the same probabilities. The state of the system at time T is thus fully specified by the number n_T of toppling sites. This is essentially a Galton-Watson branching process, and is discussed in the introductory monograph by Harris [4].

Following the notation in [3], let C_n correspond to the probability that a toppled site propogates n toppled sites at the next time step. Then, $C_0 = (1-P)^2$, $C_1 = 2P(1-P)$, and $C_2 = P^2$, where P is the probability that a site will topple if a grain of sand is added to it. Equivalently, P is the probability that the value at a site is one. Note that a site's value is zero or one, depending on its initial value, and on whether an even or odd number of sand grains have been added to it. Since grains arrive from upstream and are unaffected by the site's value, there is no preference for zero or one; at "equilibrium," after many avalanches, P should approach $\frac{1}{2}$.

For $P < \frac{1}{2}$, there is a characteristic lifetime x for the branching process that is straightforward to derive. If there are n_T toppling sites at time step T, then the expected number at the next step will be $\langle n_{T+1} \rangle = (0C_0 + 1C_1 + 2C_2)n_T = 2Pn_T$. Thus, $\langle n_T \rangle \sim (2P)^T = e^{-T/x}$, with characteristic lifetime $x = -1/\ln(2P)$.

Alstrøm [3] considers the critical value $P = \frac{1}{2}$, and compares this branching model to self-organized criticality. But when there is a concentration $p > 0$ of hole defects, criticality is lost, and the analysis simplifies. In this case, $P_{\text{eff}} = (1-p)P$ since a site has a chance of toppling only if it is not a hole. At equilibrium, the characteristic lifetime is therefore given by $x = -1/\ln(1-p)$. For small p, this approaches a power law $x \approx p^{-1}$. This differs from the numerically observed scaling by a constant factor, but the exponent predicted by the branching model ($\mu_x = 1$) agrees with the numerical simulations.

In Fig. 1, a plot of $\ln x$ versus $\ln p$ shows both the

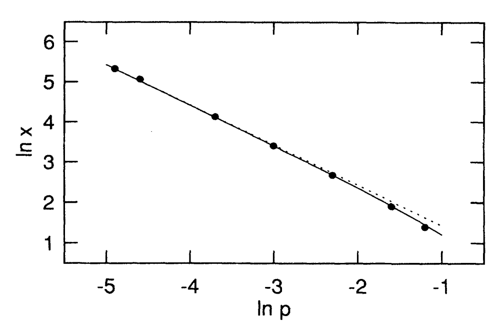

FIG. 1. Scaling of characteristic duration x as a function of the concentration p of holes. Solid circles (•) are from the numerical experiments in [2]. The solid line (—) is the prediction $x = -k/\ln(1-p)$ which is the mean-field value multiplied by a constant factor $k = 1.54$ chosen to fit the data. The dotted line (···) is the power law $x = -k/p$.

experimental values reported in [2] (from Fig. 2 of that paper), and the prediction of the mean-field approximation, after multiplication by a constant factor of ~ 1. The theory describes not only the scaling exponent μ_x, but also the deviation from power-law behavior observed at larger p.

It should be emphasized that the mean field is a heuristic theory, and does not predict the coefficient of the scaling or the critical density $p^* \approx 0.295$ above which the largest possible avalanche becomes finite [2]. Also, direct application of this model to obtain μ_m gives a value of 2 [cf. Eq. (5) of [3]]. However, in the "pure" ($p = 0$) sandpile [1], the scaling of mass m with lifetime x was shown to follow an exact power law: $m \sim x^{3/2}$. Since $x \sim p^{-\mu_x}$ and $m \sim p^{-\mu_m}$, one expects $m \sim x^{\mu_m/\mu_x}$ and so $\mu_m = \frac{3}{2}$, in fair agreement with the reported value of 1.52 ± 0.01.

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[4] T. E. Harris, *The Theory of Branching Processes* (Prentice-Hall, Englewood Cliffs, 1963); reprinted by Dover, New York, 1989.